On Asymptotic Weil-Petersson Geometry of Teichmüller Space of Riemann Surfaces

Zheng Huang

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1 Introduction

A remarkable property of hyperbolic surfaces is that many distinct complex structures, or equivalently hyperbolic structures, can be introduced

on a surface. The moduli problem of Riemann asks how many distinct complex structures can exist on a closed surface, and this problem has been developed into the modern theory of Teichmüller space.

In this paper, we assume Σ is a smooth, closed Riemann surface of genus g, with n punctures and $3g-3+n>1$. Teichmüller space $\mathcal{T}_{g,n}$ is the space of hyperbolic metrics (with constant curvature -1) on Σ, where two hyperbolic metrics σ and ρ are equivalent if there is a biholomorphic map between (Σ,σ) and (Σ,ρ) in the homotopy class of the identity map.

Teichmüller space $\mathcal{T}_{g,n}$ has its own natural complex structure ([1]): $\mathcal{T}_{g,n}$ is a complex manifold of complex dimension $3g-3+n>1$, the cotangent space at Σ is identified with $QD(\Sigma)$, the space of holomorphic quadratic differentials; and the tangent space at Σ is identified with the space of so-called harmonic Beltrami differentials.

The Weil-Petersson metric on Teichmüller space is naturally defined by duality from the L^2 inner product on $QD(\Sigma)$. This metric is considered as one of the natural metrics on Teichmüller space. Every Weil-Petersson isometry of Teichmüller space is induced by an element of the extended mapping class group when $3g-3+n>1$ and $(g,n)\neq(1,2)$ ([14]). And the Weil-Petersson metric has many interesting geometric properties: for example, it is a Riemannian metric with negative sectional curvature ([25] [21]); yet it is incomplete, since not every geodesic can be extended indefinitely, the surface Σ developes a node when a geodesic cannot be further extended ([13] [5] [24]); this metric is Kähler ([1]), and there is a negative upper bound $\frac{-1}{2\pi(g-1)}$, which only depends on the topology of the surface, for the holomorphic sectional curvature and Ricci curvature ([25]); however, there are no negative upper bounds for the sectional curvature ([10]).

Our goal in this paper is to investigate the asymptotic geometry of the Weil-Petersson metric on Teichmüller space and give an estimate on the upper and lower bounds for the Weil-Petersson sectional curvature at any point in Teichmüller space, purely in terms of the length of the shortest geodesic on the surface:

Theorem 1.1. *Let l be the length of the shortest geodesic on closed surface Σ, and K be the Weil-Petersson sectional curvature of Teichmüller space $\mathcal{T}$, assuming $dim_C\mathcal{T}>1$, there exists a constant $C>0$ such that*

$$-(Cl)^{-1}\leq K\leq -Cl.$$

Moreover, there are tangent planes with the Weil-Petersson curvatures

of the orders $O(l)$ and comparable to l^{-1}, and hence the Weil-Petersson sectional curvature has neither negative upper bound, nor lower bound.

We will firstly prove a result on asymptotic holomorphic sectional curvature. Our estimate indicates that the holomorphic sectional curvature tends to negative infinity when a core geodesic on the surface is performed infinitesimal twists and length shrinking. More specifically, we show that

Theorem 1.2. *If the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, then there is no negative lower bound for the holomorphic sectional curvature of the Weil-Petersson metric. Moreover, let l be the length of the shortest geodesic along a path to the frontier of Teichmüller space, then there exists a sequence of tangent planes with Weil-Petersson holomorphic sectional curvature of the order comparable to l^{-1}.*

We realized that one of the difficulties in estimating curvatures is working with the operator $D = -2(\Delta - 2)^{-1}$, which appears in Tromba-Wolpert's curvature formula. As pointed out in ([22]), there is a natural correspondence of the operator D and local variations of the energy of a harmonic map between surfaces. By investigating harmonic maps from a nearly noded surface to nearby hyperbolic structures, in [10], we showed that even though the sectional curvatures are negative, they are not staying away from zero. More specifically, we detected an asymptotically flat tangent plane, denoted by Ω_l, spanned by two Beltrami differentials $\dot{\mu}_0$ and $\dot{\mu}_1$, resulting from pinching two independent core geodesics on the surface.

Theorem 1 of [10]. *If the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, then the Weil-Petersson sectional curvature is not pinched from above by any negative constant. Moreover, the Weil-Petersson sectional curvature of Ω_l is of the order $O(l)$.*

In theorem 1.2, we are detecting a tangent plane, spanned by Beltrami differentials $\dot{\mu}_0$ and $i\dot{\mu}_0$, whose curvature is asymptotically negative infinity. Following a suggestion of Scott Wolpert, we consider a family of tangent planes Ω_l', spanned by $i\dot{\mu}_0$ and $\dot{\mu}_1$, and find that

Theorem 1.3. *Ω_l' is asymptotically flat, i.e., its Weil-Petersson curvature is of the order $O(l)$.*

In other words, we are detecting another asymptotically flat tangent plane, spanned by $i\dot{\mu}_0$ and $\dot{\mu}_1$. This asymptotic flatness results from

pinching two nonhomotopic closed geodesics on the surface while performing infinitesimal twists on one of them. Together with theorem 1 of ([10]), this suggests an asymptotic product structure of Weil-Petersson metric, as pointed out by Wolpert ([26]). Similarly, we also show that:

Theorem 1.4. *The plane Ω_l'' spanned by Beltrami differentials $i\dot{\mu}_0$ and $i\dot{\mu}_1$ is asymptotically flat with respect to the Weil-Petersson metric, and its Weil-Petersson sectional curvature is of the order $O(l)$.*

We notice that theorems 1.3, 1.4 and theorem 1 of ([10]) are theorems concerning asymptotic flatness, and the path we take towards the frontier of Teichmüller space (see definition of the frontier of Teichmüller space in §2.1) is to pinch two short independent geodesics on the surface. We can also pinch just one simple closed curve on the surface to take a path towards the frontier space. There is also a phenomenon of asymptotic flatness when a separating geodesic on the surface is pinched. When Σ is a closed surface with genus at least two and γ_0 is a separating short geodesic on Σ with length l. Let γ_2 and γ_3 be two closed geodesics on Σ which have fixed length $l_0 >> l$, and these two geodesics lie on different sides of the shrinking curve γ_0. We can define a tangent plane Ω_l''', spanned by Beltrami differentials $\dot{\mu}_2$ and $\dot{\mu}_3$, where $\dot{\mu}_2$ and $\dot{\mu}_3$ are obtained from infinitesimal twists about curves γ_2 and γ_3, respectively.

Theorem 1.5. *This plane Ω_l''' is asymptotically flat with respect to the Weil-Petersson metric, moreover, its Weil-Petersson sectional curvature is of the order $O(l)$.*

Here is the more detailed content of this paper. We give the necessary background in section 2. Section 3 is devoted to proving theorem 1.2. The discussion of this purpose is broken into subsections: in §3.1, we study a harmonic mapping problem between hyperbolic cylinders. The family of rotationally symmetric harmonic maps will be an approximation to the actual family of harmonic maps restricted on the pinching neighborhood. We will describe so called "model case one" in §3.2, namely, we pinch one core geodesic of a cylinder into a point, and perform infinitesimal twists about this geodesic, and study the asymptotic behavior of the harmonic maps between cylinders. We will establish the estimates of terms in the curvature formula in model case one; in §3.3, we construct families of maps which have small tension, and are close to the harmonic maps resulting from pinching and twisting process in §3.2; finally in §3.4, we prove theorem 1.2 based on the estimates in §3.2 and

the construction in §3.3. In section 4 we study the asymptotic flatness of the Weil-Petersson metric with two nonhomotopic geodesics on the surface are shrinking. We describe model case two in §4.1, estimate curvature terms in §4.2, and prove theorems 1.3 and 1.4 in §4.3. The aim of section 5 is to prove curvature bounds for the Weil-Petersson metric. In §5.1, we consider the asymptotic flatness when only one separating geodesic on the surface is pinched, and prove theorem 1.5; We prove theorem 1.1 in §5.2.

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2 Harmonic Maps and Teichmüller Space

We will give some background in this section.

2.1 Teichmüller Space of Riemann Surfaces

Recall that Σ is a fixed, oriented, smooth surface of genus $g \geq 1$, and $n \geq 0$ punctures where $3g - 3 + n > 1$. We denote hyperbolic metrics on the surface Σ by $\sigma|dz|^2$ and $\rho|dw|^2$, where z and w are conformal coordinates on Σ. On $(\Sigma, \sigma|dz|^2)$, we denote

$$\Delta = \tfrac{4}{\sigma}\tfrac{\partial^2}{\partial z \partial \bar{z}},$$
$$K(\rho) = -\tfrac{2}{\rho}\tfrac{\partial^2}{\partial w \partial \bar{w}} log\rho,$$
$$K(\sigma) = -\tfrac{2}{\sigma}\tfrac{\partial^2}{\partial z \partial \bar{z}} log\sigma,$$

where Δ is the Laplacian, and $K(\rho)$, $K(\sigma)$ are curvatures of the metrics ρ and σ, respectively.

By the uniformization theorem, the set of all similarly oriented hyperbolic structures M_{-1} can be identified with the set of all conformal (or complex) structures on Σ with the given orientation. And Teichmüller space $\mathcal{T}$ is defined to be the quotient space

$$\mathcal{T} = M_{-1}/Diff_0(\Sigma)$$

The moduli space of Riemann surfaces admits the Deligne-Mumford compactification ([16]), and any element of the compactification divisor can be thought of as a Riemann surface with nodes, a connected

complex space where points have neighborhoods complex isomorphic to either $\{|z| < \varepsilon\}$ (regular points) or $\{zw = 0; |z|, |w| < \varepsilon\}$ (nodes). We can think of noded surfaces arising as elements of the compactification divisor through a pinching process: fix a family of simple closed curves on the surface Σ such that each component of the complement of the curves has negative Euler characteristic. Topologically, the noded surface is the result of identifying each curve to the node ([3]).

Teichmüller space $\mathcal{T}$ is a complex manifold when $3g - 3 + n > 1$, and the cotangent space at a point $\Sigma \in \mathcal{T}$ is the space of holomorphic quadratic differentials Φdz^2 on Σ ([1]). The Weil-Petersson metric on $\mathcal{T}$ is defined on $QD(\Sigma) \cong T^*_\Sigma \mathcal{T}$ by the L^2-norm:

$$||\phi||^2 = \int_\Sigma \frac{|\phi|^2}{\sigma} dz d\bar{z}$$

where $\sigma|dz|^2$ is the hyperbolic metric on Σ. By duality, we obtain a Riemannian metric on the tangent space to $\mathcal{T}$.

Teichmüller space with Weil-Petersson metric is not complete ([5] [24]). The incompleteness is caused by pinching of at least one short geodesic on the surface. For the surface with genus at least two, we define $\bar{\mathcal{T}}$ to be the Weil-Petersson completion of $\mathcal{T}$, and denote $\partial\mathcal{T}$ as the frontier set $\bar{\mathcal{T}} \backslash \mathcal{T}$. Hence as shown in ([13]), every point in $\partial\mathcal{T}$ represents a noded surface, i.e., the frontier set $\partial\mathcal{T}$ consists of a union of lower dimensional Teichmüller spaces, each such space consists of topologically reduced Riemann surfaces (noded surfaces), obtained by pinching nontrivial geodesics on the surface.

The curvature tensor of the Weil-Petersson metric is given by ([25])

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = (\int_\Sigma D(\dot{\mu}_\alpha \dot{\bar{\mu}}_\beta \dot{\mu}_\gamma \dot{\bar{\mu}}_\delta dA) + (\int_\Sigma D(\dot{\mu}_\alpha \dot{\bar{\mu}}_\delta) \dot{\mu}_\gamma \dot{\bar{\mu}}_\beta dA)$$

where dA is the area element and $\dot{\mu}$'s are infinitesimal Beltrami differentials. Here the operator $D = -2(\Delta - 2)^{-1}$. It is known that the operator D is a self-adjoint compact integral operator with a positive kernel, and it is the identity on constant functions.

Then the curvature of Ω is then given by R/Π, where ([25])

$$R = R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}}$$

and

$$\begin{aligned} \Pi &= 4 < \dot{\mu}_0, \dot{\mu}_0 >< \dot{\mu}_1, \dot{\mu}_1 > -2| < \dot{\mu}_0, \dot{\mu}_1 > |^2 - 2Re(< \dot{\mu}_0, \dot{\mu}_1 >)^2 \\ &= 4 < \dot{\mu}_0, \dot{\mu}_0 >< \dot{\mu}_1, \dot{\mu}_1 > -4| < \dot{\mu}_0, \dot{\mu}_1 > |^2 \end{aligned}$$

It is known that the Weil-Petersson sectional curvature, holomorphic sectional curvature and Ricci curvature are all negative ([25] [21]).

2.2 Harmonic Maps and Local Variations

Our main method is to look at families of harmonic maps between degenerating Riemann surfaces. The method of harmonic maps has been intensively studied as an important computational tool in understanding the geometry of Teichmüller space. In particular, the second variation of the energy of the harmonic map $w = w(\sigma, \rho)$ with respect to the domain structure σ (or image structure ρ) at $\sigma = \rho$ yields the Weil-Petersson metric on $\mathcal{T}$ ([21], [22]), and one can also re-establish Tromba-Wolpert's curvature tensor formula of the Weil-Petersson metric from this method ([11], [22]).

For a Lipschitz map $w : (\Sigma, \sigma|dz|^2) \to (\Sigma, \rho|dw|^2)$, we define the energy density of w at a point to be

$$e(w; \sigma, \rho) = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2 + \frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2$$

and the total energy $E(w; \sigma, \rho) = \int_\Sigma e(w; \sigma, \rho)\sigma dz d\bar{z}$.

A harmonic map is a critical point of the energy fuctional $E(w; \sigma, \rho)$; it satisfies the Euler-Lagrange equation, namely,

$$w_{z\bar{z}} + \frac{\rho_w}{\rho} w_z w_{\bar{z}} = 0.$$

The Euler-Lagrange equation for the energy is the condition for the vanishing of the *tension*, which is, in local coordinates,

$$\tau(w) = \Delta w^\gamma +^N \Gamma^\gamma_{\alpha\beta} w^\alpha_i w^\beta_j = 0$$

It is fundamental([2] [7] [9] [19] [18]) that given σ, ρ, there exists a unique harmonic map $w : (\Sigma, \sigma) \to (\Sigma, \rho)$ homotopic to the identity of Σ, and this map is in fact a diffeomorphism. Naturally associated to a harmonic map $w : (\Sigma, \sigma|dz|^2) \to (\Sigma, \rho|dw|^2)$ is a quadratic differential $\Phi(\sigma, \rho)dz^2$, which is holomorphic with respect to the conformal structure of σ. This association of a quadratic differential to a conformal structure then defines a map $\Phi : \mathcal{T} \to QD(\Sigma)$ from Teichmüller space $\mathcal{T}$ to the space of holomorphic quadratic differentials $QD(\Sigma)$. This map Φ is a homeomorphism ([22]).

Thus we have a holomorphic quadratic differential $\Phi dz^2 = \rho w_z \bar{w}_z dz^2$, and evidently

$$\Phi = 0 \Leftrightarrow w \text{ is conformal} \Leftrightarrow \sigma = \rho.$$

where $\sigma = \rho$ means that (Σ, σ) and (Σ, ρ) are the same point in Teichmüller space $\mathcal{T}$. Also note that the map w can be extended to surfaces with finitely many punctures.

We define two auxiliary functions as following:

$$\mathcal{H} = \mathcal{H}(\sigma,\rho) = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2$$
$$\mathcal{L} = \mathcal{L}(\sigma,\rho) = \frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2$$

The Euler-Lagrange equation gives that

$$\Delta log\mathcal{H} = -2K(\rho)\mathcal{H} + 2K(\rho)\mathcal{L} + 2K(\sigma).$$

When we restrict ourselves to the situation when $K(\sigma) = K(\rho) = -1$, we will have the following facts ([22]):

- The energy density is $e = \mathcal{H} + \mathcal{L}$;
- The Jacobian is $\mathcal{H} - \mathcal{L}$;
- $\mathcal{H} > 0$;
- The Beltrami differential $\mu = \frac{w_{\bar{z}}}{w_z} = \frac{\bar{\Phi}}{\sigma\mathcal{H}}$;
- $\Delta log\mathcal{H} = 2\mathcal{H} - 2\mathcal{L} - 2$, where $\mathcal{H} \neq 0$; and $\Delta log\mathcal{L} = 2\mathcal{L} - 2\mathcal{H} - 2$, where $\mathcal{H} \neq 0$.

Now we consider a family of harmonic maps $w(t)$ for t small, where $w(0) = \text{id}$, the identity map. Denote by $\Phi(t)$ the family of Hopf differentials determined by $w(t)$. We rewrite $\Delta log\mathcal{H} = 2\mathcal{H} - 2\mathcal{L} - 2$ as

$$\Delta log\mathcal{H}(t) = 2\mathcal{H}(t) - \frac{2|\Phi(t)|^2}{\sigma^2\mathcal{H}(t)} - 2$$

The maximum principle will force all the odd order t-derivatives of the holomorphic energy $\mathcal{H}(t)$ to vanish, since the above equation only depends on the modulus of $\Phi(t)$ and not on its argument([22]), and $\mathcal{H}(t)$ is real-analytic in t ([23]).

For the family of harmonic maps $w(t)$, the domain hyperbolic structure is fixed and the target metric is changing. Wolf computed the t-derivative of various geometric quantities associated with this family $w(t)$, and we collect these local variational formulas into:

Lemma 2.1. *([22]) For the above notations, we have*

- $\mathcal{H}(t) \geq 1$, *and* $\mathcal{H}(t) \equiv 1 \Leftrightarrow t = 0$*;*
- $\dot{\mathcal{H}}(t) = \partial/\partial t^\alpha|_0\ \mathcal{H}(t) = 0$*;*
- $\dot{\mu} = \partial/\partial t^\alpha|_0\mu(t) = \bar{\Phi}_\alpha/\sigma$*;*
- $\ddot{\mathcal{H}}(t) = \frac{\partial^2}{\partial t^\alpha \partial t^\beta}|_0\mathcal{H}(t) = D\frac{\Phi_\alpha\Phi_{\bar{\beta}}}{\sigma^2}$.

With this lemma, and recall that $D = -2(\Delta - 2)^{-1}$, we obtain a partial differential equation about $\ddot{\mathcal{H}}(t)$ which will play an important role in the proof of our theorems:

$$(\Delta - 2)(\ddot{\mathcal{H}}(t)) = -2\frac{\Phi_\alpha \Phi_{\bar{\beta}}}{\sigma^2}$$

3 Holomorphic Sectional Curvature

In this section, we will focus on estimating the Weil-Petersson holomorphic sectional curvature, and prove theorem 1.2, namely,

Theorem 1.2. *If the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, then there is no negative lower bound for the holomorphic sectional curvature of the Weil-Petersson metric. Moreover, let l be the length of the shortest geodesic along a path to a boundary point in Teichmüller space, then there exists a sequence of tangent planes with Weil-Petersson holomorphic sectional curvature of the order comparable to l^{-1}.*

We organize this section into following subsections. In §3.1, we will study harmonic maps between hyperbolic cylinders, i.e., we consider a family of pertubations of the identity map between two cylinders; in §3.2, we estimate terms in the curvature formula in the model case one where the surface is a long cylinder; in §3.3, we construct a family of $C^{2,\alpha}$ maps between surfaces and show that the constructed maps have small tension and are close to the harmonic maps we obtain from the pinching and twisting process; in §3.4, we adapt the estimates in §3.2 and §3.3 to prove theorem 1.2.

For the sake of simplicity of exposition, we assume that our surface has no punctures. We will discuss the case when finitely many punctures are allowed in §3.5.

3.1 Harmonic Maps between Cylinders

In this subsection, we consider the asymptotics of harmonic maps, as pertubations of the identity map, between two hyperbolic cylinders. In particular, using the same notation in [10], we study the boundary value problem of harmonically mapping the cylinder

$$M = [l^{-1}sin^{-1}(l), \pi l^{-1} - l^{-1}sin^{-1}(l)] \times [0, 1]$$

with boundary identification

$$[\frac{sin^{-1}(l)}{l}, \frac{\pi}{l} - \frac{sin^{-1}(l)}{l}] \times \{0\} = [\frac{sin^{-1}(l)}{l}, \frac{\pi}{l} - \frac{sin^{-1}(l)}{l}] \times \{1\}$$

where the hyperbolic length element on M is $lcsc(lx)|dz|$, to the cylinder

$$N = [L^{-1}sin^{-1}(L), \pi L^{-1} - L^{-1}sin^{-1}(L)] \times [0,1]$$

with boundary identification

$$[\frac{sin^{-1}(L)}{L}, \frac{\pi}{L} - \frac{sin^{-1}(L)}{L}] \times \{0\} = [\frac{sin^{-1}(L)}{L}, \frac{\pi}{L} - \frac{sin^{-1}(L)}{L}] \times \{1\}$$

where the hyperbolic length element on N is $Lcsc(Lu)|dw|$. Here l and L are the lengths of the simple closed core geodesics in the corresponding cylinders.

Let $w = u + iv$ be this harmonic map between cylinders M and N, where

$$u(l, L; x, y) = u(l, L; x), v(x, y) = y.$$

The Euler-Lagrange equation becomes

$$u'' = Lcot(Lu)(u'^2 - 1).$$

with boundary conditions $u(\frac{sin^{-1}(l)}{l}) = \frac{sin^{-1}(L)}{L}$ and $u(\frac{\pi}{2l}) = \frac{\pi}{2L}$. Note that both M and N admit an anti-isometric reflection about the curves $\{\frac{\pi}{2l}\} \times [0,1]$ and $\{\frac{\pi}{2L}\} \times [0,1]$.

Since the quadratic differential $\Phi = \rho w_z \bar{w}_z = \frac{1}{4}L^2csc^2(Lu)(u'^2 - 1)$ is holomorphic in M, we have

$$0 = \frac{\partial}{\partial \bar{z}}(\rho^2 w_z \bar{w}_z) = \frac{\partial}{\partial x}(\frac{1}{8}L^2csc^2(Lu)(u'^2 - 1))$$

Therefore $L^2csc^2(Lu)(u'^2 - 1) = c_0(l, L)$, where $c_0(l, L)$ is independent of x, and $c_0(l, l) = 0$ since $u(x)$ is the identity map when $L = l$.

So we have

$$u' = \sqrt{1 + c_0(l, L)L^{-2}sin^2(Lu)}$$

with boundary conditions $u(\frac{sin^{-1}(l)}{l}) = \frac{sin^{-1}(L)}{L}$ and $u(\frac{\pi}{2l}) = \frac{\pi}{2L}$. So the solution to the Euler-Lagrange equation can be derived from the equation

$$\int_{L^{-1}sin^{-1}L}^{u} \frac{dv}{\sqrt{1 + c_0(l, L)L^2 sin^2(Lv)}} = x - l^{-1}sin^{-1}(l)$$

with $c_0(l,L)$ chosen such that $\int_{L^{-1}sin^{-1}L}^{\frac{\pi}{2L}} \frac{dv}{\sqrt{1+c_0(l,L)L^2 sin^2(Lv)}} = \frac{\pi}{2l} - l^{-1}sin^{-1}l$.

It is not hard to show that when $l \to 0$, the solution $u(l,L;x)$ converges to a solution $u(L;x)$ to the "noded" problem, i.e., $M = [1, +\infty)$, where we require $u(L;1) = L^{-1}sin^{-1}(L)$ and $\lim_{x\to+\infty} u(L;x) = \frac{\pi}{2L}$.

This "noded" problem has the explicit solution as following ([23])

$$u(L;x) = L^{-1}sin^{-1}\{\frac{1-\frac{(1-L)}{(1+L)}e^{2L(1-x)}}{1+\frac{(1-L)}{(1+L)}e^{2L(1-x)}}\}$$

with the holomorphic energy

$$\mathcal{H}_0(L;x) = \frac{L^2x^2}{4}[\frac{1+\sqrt{\frac{(1-L)}{(1+L)}}e^{L(1-x)}}{1-\sqrt{\frac{(1-L)}{(1+L)}}e^{L(1-x)}}]^2$$

3.2 Model Case One

Consider the surface Σ which is developing a node, i.e., we are pinching one short closed geodesic γ_0 on Σ to a point p_0. We denote M_0 its pinching neighborhood, i.e., a cylinder described as M in §3.1 centered at γ_0. In our model case one, the surface is this cylinder M_0 with core geodesic γ_0.

We define $M(l,\theta)$ be the surface with two of the Fenchel-Nielsen coordinates, namely, the hyperbolic length of γ_0 being l and the twisting angle of γ_0 being θ. We obtain a point $M(l) = M(l,0)$ in Teichmüller space $\mathcal{T}_g$. Note that as l tends to zero, the surface is developing a node. Fix $M(l)$, we vary the length of γ_0 into length $L = L(t)$, where $L(0) = l$. Thus we obtain a family of harmonic maps $W_0(t) : M(l) \to M(L(t),0) = M(L(t))$. The t-derivative of $\mu_0(t)$, the Beltrami differential of the the family $W_0(t)$, at $t = 0$ represents a tangent vector, $\dot{\mu}_0$, of Teichmüller space $\mathcal{T}_g$ at $M(l)$; And a tangent vector $i\dot{\mu}_0$ is obtained by performing infinitesimal twists $\theta(t)$ on the shrinking curve γ_0, where $\theta(0) = 0$, since $\frac{d}{d\theta}(e^{i\theta}\dot{\mu}_0)|_{\theta=0} = i\dot{\mu}_0$. Hence these two tangent vectors $\dot{\mu}_0$ and $i\dot{\mu}_0$ will span a two dimensional subspace of the tangent space $T_{M(l)}\mathcal{T}_g$ to $\mathcal{T}_g$, therefore we obtain a family, Ω_l^*, of tangent planes.

Proposition 3.1. *The Weil-Petersson holomorphic sectional curvatures of Ω_l^* is of the order comparable to l^{-1}. Hence the Weil-Petersson curvatures tend to negative infinity as l tends to zero.*

It is easy to see that Proposition 3.1 implies our theorem 1.2.

We denote $\phi_0(t)$ as the Hopf differential corresponding to the cylinder map $w_0(t)$ in M_0. Here $w_0(t) : M_0(l) \to M_0(L(t))$ is a family of harmonic maps described as $w = u(x) + iy$ in §3.1. In other regions of the surface, we abuse our notations a little bit and still use ϕ_0 to denote the Hopf differential corresponding to harmonic map $W_0(t)$. We also denote μ_0 as the corresponding Beltrami differential to ϕ_0.

We denote $a = a(l) = l^{-1}sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1}sin^{-1}(l)$. And in this paper, $A \sim B$ means $A/C \leq B \leq CA$ for some constant $C > 0$.

So in M_0, recalling that $c_0(l, L(t)) = L^2csc^2(Lu)(u'^2 - 1)$ is independent of x, we can choose $L(t)$ so that $\frac{d}{dt}|_{t=0}c_0(t) = \frac{d}{dL}|_{L=l}c_0(l, L) = 4$. Thus we can assume that $\dot{\phi}_0 = \frac{d}{dt}|_{t=0}(\frac{1}{4}c_0(t)) = 1$ in M_0. We notice here $\dot{c}_0 = \frac{d}{dt}|_{t=0}$ is never zero for all positive l, otherwise, we would have $\dot{w}_z = 0$ as $\dot{\phi} = \sigma\dot{w}_z$ and hence w is a constant map by rotational invariance of the map.

Note that most of the mass of $|\phi_0|$ resides in the thin part associated to γ_0, near where the core geodesic γ_0 is pinched, and that is why the estimate in the long cylinder is critical in our calculation.

We will now estimate all terms in the curvature formula in this model case one where we consider the surface as a long cylinder M_0. In M_0, the corresponding Beltrami differential is

$$\dot{\mu}_0 = \frac{d}{dt}|_{t=0}(\frac{w_{\bar{z}}}{w_z}) = \dot{\bar{\phi}}_0/\sigma$$

and $|\dot{\mu}_0|^2|_{M_0} = |i\dot{\mu}_0|^2_{M_0} = |\dot{\bar{\phi}}_0/\sigma|^2|_{M_0} = l^{-4}sin^4(lx)$.

The Weil-Petersson holomorphic sectional curvature is given by the quotient $-R_{0\bar{0}0\bar{0}}/|\dot{\mu}_0|^4$. And we will estimate $1/|\dot{\mu}_0|^4$ and $|R_{0\bar{0}0\bar{0}}|$ in the cylinder M_0 in the next two lemmas.

Lemma 3.2. $1/|\dot{\mu}_0|^4 \sim l^6$.

Proof. To show this lemma, we use $|\dot{\mu}_0|^2|_{M_0} = l^{-4}sin^4(lx)$, and noticing that $a = a(l) = l^{-1}sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1}sin^{-1}(l)$, then we

have

$$\begin{aligned} < \dot{\mu}_0, \dot{\mu}_0 > |_{M_0} &= \int_{M_0} |\dot{\mu}_0|^2 \sigma dxdy \\ &= \int_0^1 \int_a^b |\dot{\mu}_0|^2 \sigma dxdy \\ &= \int_0^1 \int_a^b l^{-2} sin^2 lx dxdy \\ &= \frac{\pi}{2} l^{-3} - l^{-3} sin^{-1}(l) \\ &\sim l^{-3} \end{aligned}$$

Note that $|\dot{\mu}_0|^4 \sim (< \dot{\mu}_0, \dot{\mu}_0 > |_{M_0})^2 \sim l^{-6}$, which completes the proof of Lemma 3.2. □

Now we are left to estimate $|R_{0\bar{0}0\bar{0}}| = \int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 \sigma dxdy$. The desired estimate is to establish

Lemma 3.3. $\int_{M_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 dA \sim l^{-7}$

Proof. Firstly, from Lemma 2.1, we have

$$D(|\dot{\mu}_0|^2) = -2(\Delta - 2)^{-1} \tfrac{|\dot{\phi}_0|^2}{\sigma^2}$$

We recall in M_0, the Hopf differential ϕ_0 is corresponding to the cylinder map $w_0(t) : (M_0, \sigma) \to (M_0, \rho(t))$, and $|\dot{\phi}_0| = 1$. As in §2.2 and §3.1, we write the holomorphic energy $\mathcal{H} = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2$, therefore we can write $D(|\dot{\mu}_0|^2) = \ddot{\mathcal{H}}$. Then in the cylinder M_0, we have

$$(\Delta - 2)\ddot{\mathcal{H}} = -2\frac{|\dot{\phi}_0|^2}{\sigma^2} = -2l^{-4} sin^4(lx) \tag{1}$$

A maximum principle argument implies that $\ddot{\mathcal{H}}$ is positive. This $\ddot{\mathcal{H}}$ converges to the holomorphic energy $\ddot{\mathcal{H}}_0$ in the "noded" problem (of §3.1) when x is fixed but sufficiently large in $[a, \pi/2l]$. This convergence guarantees that $\ddot{\mathcal{H}}$ is bounded on the compacta in $[a, b]$ and so we can assume that $A_1(l) = \ddot{\mathcal{H}}(a) = \ddot{\mathcal{H}}(l^{-1} sin^{-1} l) = O(1) > 0$. Then $\ddot{\mathcal{H}}(x)$ solves the following differential equation

$$(l^{-2} sin^2(lx))\ddot{\mathcal{H}}'' - 2\ddot{\mathcal{H}} = -2l^{-4} sin^4(lx) \tag{2}$$

with the conditions

$$\ddot{\mathcal{H}}(l^{-1}sin^{-1}l) = A_1(l), \ddot{\mathcal{H}}'(\pi/2l) = 0$$

Recall from §3.1 that all the odd order t-derivatives of $\mathcal{H}(t)$ vanish. Also notice that

$$J(x) = \tfrac{sin^2(lx)}{2l^4}$$

is a particular solution to equation (2). Hence we can check, by the method of reduction of the solutions, the general solution to equation (2) with the assigned conditons has the form

$$\ddot{\mathcal{H}}(l;x) = J(x) + A_2 cot(lx) + A_3(1 - lxcot(lx))$$

where coefficients $A_2 = A_2(l)$ and $A_3 = A_3(l)$ are constants independent of x and we can check, by substituting the solution into the assigned conditions, that A_2 and A_3 are of the order comparable to l^{-1}.

We compute the following

$$\begin{aligned}
\int_{M_0} \ddot{\mathcal{H}}(x)|\dot{\mu}_0|^2\sigma dxdy &= \int_0^1\int_a^b \ddot{\mathcal{H}}(x)(l^{-2}sin^2(lx))dxdy \\
&= \int_0^1\int_a^b J(x)l^{-2}sin^2(lx)dxdy \\
&+ \int_0^1\int_a^b A_2 cot(lx)l^{-2}sin^2(lx)dxdy \\
&+ \int_0^1\int_a^b A_3(1 - lxcot(lx))l^{-2}sin^2(lx)dxdy \\
&\sim l^{-7} + O(l^{-4}) + O(l^{-4}) \\
&\sim l^{-7}
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{M_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2\sigma dxdy &= \int_a^b \ddot{\mathcal{H}}|\dot{\mu}_0|^2\sigma dx \\
&\sim l^{-7} \qquad (3)
\end{aligned}$$

This completes the proof of Lemma 3.3. $\square$

3.3 Construction of Maps

In last subsection, we estimated the terms in the curvature formula in the model case one where the surface is a long cylinder. Essentially,

in the pinching neighborhood M_0, we used the cylinder map w_0 (rotationally symmetric harmonic map) instead of the actual harmonic map W_0 during the computation. Now we are in the general setting, i.e., the surface Σ is developing a node. In this subsection, we will construct a family of maps $G_0(t)$ to approximate the harmonic map $W_0(t)$, and the essential parts of this family are the identity map of the surface restricted in the non-cylindral part and the cylinder map in the pinching neighborhood. We will also show that this constructed family $G_0(t)$ is reasonably close to the harmonic maps $W_0(t)$; hence we can use the estimates we obtained in the previous subsection to the general situation.

We recall some of the notations from previous subsections. We still set M_0 to be the pinching neighborhood of the node p_0. Also $W_0(t)$ is the harmonic map corresponding to pinching γ_0 in M_0 into length $L = L(t)$, where $L(0) = l$, and twisting γ_0 with angle $\theta(t)$, where $\theta(0) = 0$. Let $w_0(t)$ be cylinder maps in model case one, we want to show that $W_0(t)$ is close to $w_0(t)$ in M_0 and is close to identity map outside of M_0.

An important feature of twisting the core geodesic γ_0 is that it will change the values of the corresponding $\ddot{\mathcal{H}}$ on the boundary of the long cylinder M_0. However, we recall from last subsection that the convergence of $\ddot{\mathcal{H}}$ to $\ddot{\mathcal{H}}_0$ in the noded problem guarantees $\ddot{\mathcal{H}}$ is bounded on the boundary of M_0. Therefore we can still use model case one to include the consideration of twisting the core geodesic, since we do not concern the boundary values of $\ddot{\mathcal{H}}$, as long as they are bounded. And in the compact region, where is far away from the shrinking and twisting curve γ_0, and changes of value $\ddot{\mathcal{H}}$ will be small, as we will see in next subsection.

We denote subsets $\Sigma_0 = \{p \in \Sigma : dist(p, \partial M_0) > 1\}$, and define the 1-tube of ∂M_0 as $B(\partial M_0, 1) = \{p \in \Sigma : dist(p, \partial M_0) \leq 1\}$, the intersection region between the long cylinder M_0 and compact region of the surface. We can construct a $C^{2,\alpha}$ map $G_0 : \Sigma \to \Sigma$ such that

$$G_0(p) = \begin{cases} w_0(t)(p), & p \in M_0 \cap \Sigma_0 \\ p, & p \in (\Sigma_0 \backslash M_0) \\ g_t(p), & p \in B(\partial M_0, 1) \end{cases}$$

where $g_t(p)$ in $B(\partial M_0, 1)$ is constructed so that it satisfies:

- $g_t(p) = p$ for $p \in \partial(\Sigma_0 \backslash (M_0 \cup B(\partial M_0, 1)))$, and $g_t(p) = w_0(t)(p)$ for $p \in \partial(M_0 \cap \Sigma_0)$;
- g_t is the identity map when $t = 0$;

- g_t is smooth and the tension of g_t is of the order $O(t)$.

We note that G_0 consists of 3 parts. It is the cylinder map of M_0 deep into the cylinder region in $M_0 \cap \Sigma_0$, the identity map in the compact region $\Sigma_0 \backslash M_0$, and a smooth map in the intersection region $B(\partial M_0, 1)$. Among three parts of the constructed map $G_0(t)$, two of them, the identity map and the cylinder map, are harmonic hence have zero tension; thus the tension of $G_0(t)$ is concentrated in $B(\partial M_0, 1)$. From §3.2, for the cylinder map $w_0 = u(x) + iy$, we have $u' = \sqrt{1 + c_0(t) L^{-2} sin^2 Lu}$, where $c_0(0) = 0, \dot{c}_0(0) = 4$. Hence for $x \in [l^{-1} sin^{-1}(l), l^{-1} sin^{-1}(l) + 1]$,

$$w_{0,z}(x, y) = \frac{1}{2}(u'(x) + 1) = \frac{1}{2}((1 + O(1)c_0(t))^{\frac{1}{2}} + 1) = 1 + O(1)t + O(t^2)$$
$$|w_{0,z}(x, y) - 1| = O(t) \to 0, (t \to 0)$$
$$|w_{0,z\bar{z}}(x, y)| = |\tfrac{1}{4} u''(x)| = O(|Lcot(Lu)(u'^2 - 1)|) = O(t)$$

Thus we can require that $|g_{t,z}| \le C_2 t$ and $|g_{t,z\bar{z}}| \le C_2 t$, and the constant $C_2 = C_2(t, l)$ is bounded in both t and l since the coefficient of t for $c_0(t)$ is bounded for small t and small l. With the local formula of the tension in §2.2, we have $\tau(G_0(t))$, the tension of $G_0(t)$ is of the order $O(t)$. Note that these constructed maps $G_0(t)$ are not necessarily harmonic.

Now we are about to compare the constructed family $G_0(t)$ and the family of harmonic maps $W_0(t)$. To do this, we consider the following function $Q_0 = cosh(dist(W_0, G_0)) - 1$.

Lemma 3.4. *$dist(W_0, G_0) \le C_3 t$ in $B(\partial M_0, 1)$, where the constant $C_3 = C_3(t, l)$ is bounded for small t and l.*

Proof. First, we claim that Q_0 is a C^2 function. Notice that both the harmonic map $W_0(t)$ and the constructed map $G_0(t)$ are the identity map when $t = 0$, and both families vary smoothly without changing homotopy type in t for sufficiently small $|t|$ ([6]). For all $l > 0$, and for any $\varepsilon > 0$, there exists a δ such that for $|t| < \delta$, we have $|W_0(t) - W_0(0)| < \frac{\varepsilon}{2}$ and $|G_0(t) - G_0(0)| < \frac{\varepsilon}{2}$. Therefore the triangular inequality implies that $|W_0(t) - G_0(t)| < \varepsilon$. Since l is positive, the Collar Theorem ([4]) implies that the surface has positive injectivity radius r bounded below, and we choose our $\varepsilon << r$, then Q_0 is well defined and smooth.

We follow an argument in [8]. For any unit $v \in T^1(B(\partial M_0, 1))$, the map G_0 satifies the inequality $|\|dG_0(v)\| - 1| = O(t)$, hence $|dG_0(v)|^2 > 1 - \varepsilon_0$ where $\varepsilon_0 = O(t)$, then we find that for any $x \in \Sigma$,

$$\begin{aligned} \Delta Q_0 \quad &\ge \quad min\{|dG_0(v)|^2 : dG_0(v) \perp \gamma_x\} Q \\ &- \quad < \tau(G_0), \bigtriangledown d(\bullet, W_0)|_{G_0(x)} > sinh(dist(W_0, G_0)) \qquad (4) \end{aligned}$$

where γ_x is the geodesic joining $G_0(x)$ to $W_0(x)$ with initial tangent vector $-\bigtriangledown d(\bullet, W_0)|_{G_0(x)}$ and terminal tangent vector $\bigtriangledown d(G_0(x), \bullet)|_{W_0(x)}$. If $G_0(t)$ does not coincide with $W_0(t)$ on $B(\partial M_0, 1)$, we must have all maxima of $Q_0(t)$ on the interior of $B(\partial M_0, 1)$, at any such maximum, we apply the inequality $|dG_0(v)|^2 > 1 - \varepsilon_0$ to (12) to find

$$0 \geq \Delta Q_0 \quad \geq \quad (1-\varepsilon_0)Q_0 - \tau(G_0)(sinh(dist(W_0, G_0))$$

so that at a maximum of Q_0, we have

$$Q_0 \leq \tfrac{\tau(G_0) sinh dist(W_0, G_0)}{(1-\varepsilon_0)}$$

We notice that Q_0 is of the order $dist^2(W_0, G_0)$ and $sinhdist(W_0, G_0)$ is of the order $dist(W_0, G_0)$, this implies that $dist(W_0, G_0)$ is of the order $O(t)$ in $B(\partial M_0, 1)$, which completes the proof of Lemma 3.4. $\square$

Remark 3.5. *Lemma 3.4 implies that $Q_0(t)$ is of the order $O(t^2)$ in $B(\partial M_0, 1)$.*

Note that $B(\partial M_0, 1)$ contains the boundary of the cylinder $M_0 \cap \Sigma_0$, which we identify with $[a+1, b-1] \times [0,1]$, where, again, $a = a(l) = l^{-1} sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1} sin^{-1}(l)$. While in the cylinder $M_0 \cap \Sigma_0$, we have the inequality

$$\begin{aligned}
\Delta Q_0 &\geq (1-\varepsilon_0)Q_0 - \tau(G_0)(sinh(dist(W_0, G_0)) \\
&= (1-\varepsilon_0)Q_0 - \tau(G_0)(tanh(dist(W_0, G_0))(1+Q_0) \\
&= (1-\varepsilon_0 - \tau(G_0))Q_0 - \tau(G_0)(tanh(dist(W_0, G_0)) \\
&\geq 1/2 Q_0 - C_4 t^2
\end{aligned}$$

where the constant C_4 is bounded for small t and l. Therefore we find that $Q_0(z,t)$ decays rapidly in $z = (x,y)$ for x close enough to $\pi/2l$. Hence we can assume that $dist(W_0, G_0)$ is at most of order $C't$ in $[a+1, b-1]$, here $C' = C'(x,l)$ is no greater that $C_5 x^{-2}$ for $x \in [a+1, \pi/2l]$, and no greater than $C_5(\pi/l - x)^{-2}$ for $x \in [\pi/2l, b-1]$, where C_5 is bounded for small t and l. Both maps W_0 and G_0 are harmonic in $M_0 \cap \Sigma_0$, so they are also C^1 close ([6]), i.e. we have $|W_{0,\bar{z}} - G_{0,\bar{z}}| \leq C_5 x^{-2} t$ for small t and l, when $x \in [a+1, \pi/2l]$. Thus we see that $|\dot{W}_{0,\bar{z}} - \dot{G}_{0,\bar{z}}| = C_5 x^{-3}$, for $x \in [a+1, \pi/2l]$. Also, $|\dot{W}_{0,\bar{z}} - \dot{G}_{0,\bar{z}}| = C_5(\pi/l - x)^{-3}$, for $x \in [\pi/2l, b-1]$.

As before we denote $\phi_0(t)$ as the family of Hopf differentials corresponding to the family of harmonic maps $W_0(t)$. We also denote μ_0 as the corresponding Beltrami differentials in $M_0 \cap \Sigma_0$. Write $\phi_0^G =$

$\rho G_{0,z}(t)\bar{G}_{0,z}(t)$. Notice that in $M_0 \cap \Sigma_0$, map G_0 is the cylinder map hence harmonic, so ϕ_0^G is the Hopf differential corresponding to G_0. When $t = 0$ we have $W_0 = G_0 =$ identity and $\rho = \sigma$, hence we can differentiate $\phi_0^G = \rho G_{0,z}(t)\bar{G}_{0,z}(t)$ in t at $t = 0$, and find that $|\dot{\phi}_0^G - \dot{\phi}_{W_0}| = \sigma|\dot{W}_{0,\bar{z}} - \dot{G}_{0,\bar{z}}| \leq C_5 l^2 x^{-3} csc^2(lx) = O(1)$ for $x \in [a+1, \pi/2l]$, approaching zero as x tends to $\pi/2l$, and $|\dot{\phi}_0^G - \dot{\phi}_{W_0}| \leq C_5 l^2(\pi/l - x)^{-3} csc^2(lx) = O(1)$ for $x \in [\pi/2l, b-1]$. Therefore we have proved

Lemma 3.6. $|\dot{\phi}_0^G - \dot{\phi}_{W_0}| = O(1)$ *for* $x \in [a+1, b-1]$.

Remark 3.7. *Recall that our normalization makes* $\dot{\phi}_{W_0} = 1$ *in* $[a, b]$*, and* $\dot{\phi}_0^G$ *is actually never zero, Lemma 3.6 implies that* $\dot{\phi}_0^G$ *is comparable to* $\dot{\phi}_{W_0}$*.*

3.4 Proof of Theorem 1.2

In last subsection, we constructed a family of maps and found that these maps are reasonably close to the actual harmonic maps between surfaces. Now we are ready to adapt the estimates in the model case one to the general setting, and prove the Proposition 3.1, which will imply theorem 1.2.

Proof of Proposition 3.1. We notice that Lemma 3.2 still holds from triangle inequality and Lemma 3.6. It will be sufficient to show that Lemma 3.3 still holds, which implies desired curvature estimate immediately.

Now we are about to estimate $\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 \sigma dxdy$, which breaks into 2 integrals as follows:

$$\begin{aligned}\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 dA &= \int_{M_0 \cap \Sigma_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 dA \\ &+ \int_{\mathcal{K}} D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 dA \qquad (5)\end{aligned}$$

where $\mathcal{K}$ is the compact set disjoint from $M_0 \cap \Sigma_0$.

For the last integral, from the previous discussion, because of the convergence of the harmonic maps to the harmonic maps of "noded" problem, we have both $|\dot{\mu}_0|$ and $|\dot{\mu}_1|$ are bounded. The maximum principle implies that $D(|\dot{\mu}_0|^2) = \ddot{\mathcal{H}} \leq sup\{|\dot{\mu}_0|^2\}$. Note that $\mathcal{K}$ is compact, hence we have the second integral is of the order of $O(1)$.

From Lemma 2.1, we have $(\Delta-2)\ddot{\mathcal{H}} = -2\frac{|\dot{\phi}_0|^2}{\sigma^2}$, so we rewirte (5) as

$$\int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy \quad = \quad \int_{M_0\cap\Sigma_0} \ddot{\mathcal{H}}|\dot{\mu}_1|^2\sigma dxdy + O(1) \qquad (6)$$

Recall that $(\Delta-2)\ddot{\mathcal{H}}^G = -2\frac{|\dot{\phi}_0^G|^2}{\sigma^2}$, where $\mathcal{H}^G$ is the holomorphic energy corresponding to the model case one when the harmonic map is the cylinder map, and ϕ_0^G is the quadratic differential corresponding to the constructed map G_0. We also denote μ_0^G to be the Beltrami differential corresponding to ϕ_0^G. Also recall, from Lemma 3.6, that $|\dot{\phi}_0 - \dot{\phi}_0^G| = O(1)$ in $M_0\cap\Sigma_0$, here $O(1)$ is bounded in l for small l. So we can set some $\lambda = O(1)$ (bounded in l for small l) such that $|\dot{\phi}_0|^2 < \lambda^2|\dot{\phi}_0^G|^2$ and at the boundary of $M_0\cap\Sigma_0$ satisfies $\ddot{\mathcal{H}} < \lambda\ddot{\mathcal{H}}^G$. For example, we can take $\lambda = 1 + max_{\partial K}(\frac{\ddot{\mathcal{H}}}{\ddot{\mathcal{H}}^G}, \frac{|\dot{\phi}_0|}{|\dot{\phi}_0^G|})$, this $\lambda = O(1)$ because at $\partial\mathcal{K} = \partial(M_0\cap\Sigma_0)$, both $\ddot{\mathcal{H}}$, $\ddot{\mathcal{H}}^G$, and $\frac{|\dot{\phi}_0|}{|\dot{\phi}_0^G|}$ are bounded. Therefore, we have

$$(\Delta-2)(\ddot{\mathcal{H}} - \lambda\ddot{\mathcal{H}}^G) = 2\frac{\lambda^2|\dot{\phi}_0^G|^2 - |\dot{\phi}_0|^2}{\sigma^2} > 0$$

So $(\ddot{\mathcal{H}} - \lambda\ddot{\mathcal{H}}^G)$ is a subsolution to the differential equation $(\Delta-2)Y = 0$, both with bounded boundary conditions. It is not hard to see that in the cylindar M_0, the solutions to $(\Delta-2)Y = 0$ have the form of $Y(l;x) = B_3cot(lx) + B_4(1 - lxcot(lx))$, where constants B_3 and B_4 satisfy that $B_3 = O(l)$ and $B_4 = O(l)$. Hence in $M_0\cap\Sigma_0$, we have $\ddot{\mathcal{H}} \le \lambda\ddot{\mathcal{H}}^G + B_3cot(lx) + B_4(1 - lxcot(lx))$. Now we find that,

$$\begin{aligned}
\int_{M_0\cap\Sigma_0} \ddot{\mathcal{H}}|\dot{\mu}_0|^2 dA \;&\le\; \int_{M_0\cap\Sigma_0} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(|\dot{\mu}_0|^2)dA \\
&\le\; \int_{M_0\cap\Sigma_0} \lambda\ddot{\mathcal{H}}^G(2|\dot{\mu}_0^G|^2 + 2|\dot{\mu}_0 - \dot{\mu}_0^G|^2)dA \\
&+\; \int_{M_0\cap\Sigma_0} Y(l;x)(2|\dot{\mu}_0^G|^2 + 2|\dot{\mu}_0 - \dot{\mu}_0^G|^2)dA
\end{aligned}$$

Recalling the computation in §3.3, we have the following:

$$\begin{aligned}
\int_{M_0} (\lambda\ddot{\mathcal{H}}^G)|\dot{\mu}_0^G|^2\sigma dxdy \;&=\; O(l^{-7}) \\
\int_{M_0} Y(l,x)|\dot{\mu}_0^G|^2\sigma dxdy \;&=\; O(l^{-2}) \\
\int_{M_0\cap\Sigma_0} (\ddot{\mathcal{H}}^G + Y(l;x))(|\dot{\mu}_0 - \dot{\mu}_0^G|^2)dA \;&=\; O(l^{-7})
\end{aligned}$$

These add up to $\int_\Sigma D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2\sigma dxdy = O(l^{-7})$, with Lemma 3.2, we have the holomorphic sectional curvature is of the order $O(l^6)O(l^{-7}) = O(l^{-1})$.

Noting that $(\Delta - 2)(\ddot{\mathcal{H}} - Y(l;x)) = -2|\dot{\mu}_0|^2 < 0$, to achieve the inequality of the other direction, we find, in $M_0 \cap \Sigma_0$, that $\ddot{\mathcal{H}} \geq Y(l;x) = B_3 cot(lx) + B_4(1 - lxcot(lx))$, for some constants $B_3 \sim l$ and $B_4 \sim l$. Combining with remark 3.7 and above estimates of the integrals, we find that the holomorphic sectional curvature is actually of the order comparable to l^{-1}. This completes the proof of Proposition 3.1. Also this proves theorem 1.2. □

3.5 Surface with Punctures

In last four subsections, we proved theorem 1.2. We want to point out here that the assumption on the surface Σ being compact is not essential. In other words, our theorem 1.2, as well as other theorems in this paper, are still true when the surface Σ has finitely many punctures. Now we prove theorem 1.2 in the case of Σ being a punctured surface.

The existence of a harmonic diffeomorphism between punctured surfaces has been investigated by Wolf ([23]) and Lohkamp ([12]). In particular, Lohkamp ([12]) showed that a homeomorphism between punctured surfaces is homotopic to a unique harmonic diffeomorphism with finite energy, and the holomorphic quadratic differential corresponding to the harmonic map in the homotopy class of the identity is a bijection between Teichmüller space of punctured surfaces and the space of holomorphic quadratic differentials.

In this case, the set $\Sigma \backslash (M_0 \cap \Sigma_0)$ is no longer compact. Let $\mathcal{K}_0$ be a compact surface with finitely many punctures, and $\{\mathcal{K}_m\}$ be a compact exhaustion of $\mathcal{K}_0$. We now estimate $\int_{\mathcal{K}_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 dA$. Let $\mathcal{H}(t)$ be the holomorphic energy corresponding to the harmonic map $w(t) : (\mathcal{K}_0, \sigma) \to (\mathcal{K}_0, \rho(t))$, then $\mathcal{H}(t)$ is bounded from above and below, and has nodal limit 1 near the punctures ([23]), hence $|\dot{\mu}_0|^2$ has the order $o(1)$ near the punctures. To see this, we consider $\mathcal{K}_0$ as the union of $\mathcal{K}_m$ and disjoint union of finitely many punctured disks, each equipped hyperbolic metric $\frac{|dz|^2}{z^2 log^2 z}$. Then $|\dot{\mu}_0| = O(|zlog^2 z|) \to 0$ as z tends to the p uncture, since the quadratic differential has a pole of at most the first order. We notice that $\partial\mathcal{K}_0$ is the boundary of the cylinders, where the harmonic maps converge to a solution to the "noded" problem as $l \to 0$, hence $D(|\dot{\mu}_0|^2) = \ddot{\mathcal{H}}(t)$ is bounded on $\partial\mathcal{K}_0$. Therefore we apply Omori-Yau maximum principle ([17] [28])

to the differential equation $(\Delta - 2)\ddot{\mathcal{H}} = -2|\dot{\mu}_0|^2$ on $\mathcal{K}_0$ and obtain that $sup(D(|\dot{\mu}_0|^2)) \leq max(sup(|\dot{\mu}_0|^2), max(D(|\dot{\mu}_0|^2))|_{\partial\mathcal{K}_0}) = O(1)$. Hence we have

$$\begin{aligned}\int_{\mathcal{K}_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 dA &\leq \int_{\mathcal{K}_0} sup(|\dot{\mu}_0|^2)|\dot{\mu}_0|^2 dA \\ &\leq O(1)O(1)Vol(\mathcal{K}_0) \\ &= O(1)\end{aligned}$$

In other words, our proof carries over to the punctured case, which completes the proof of theorem 1.2.

We point out that similar argument can apply to the proof of other theorems of this paper when finitely many punctures are allowed. For this reason, we will assume in the rest of the paper that our surface has no punctures.

4 Asymptotic Flatness I: Two Curves Pinching

In this and next sections, we aim to prove theorem 1.1. Firstly we want to discuss the asymptotic flatness of the Weil-Petersson metric on Teichmüller space, when there are at most two curves on the surface are shrinking. We deal with the two curves pinching case in this section.

As indicated in §2.1, the frontier space $\partial\mathcal{T}$ of Teichmüller space is a union of lower dimensional Teichmüller spaces, each consists of noded surfaces obtained by pinching nontrival geodesics on the surface. Hence to obtain two tangent vectors near the infinity, at least one geodesic on the surface is being pinched. In §4.1, we describe model case two where the surface is a pair of cylinders; in §4.2, we prove theorem 1.3 in model case two; we give general proofs of theorems 1.3 and 1.4 in §4.3.

4.1 Model Case Two

In this subsection, following [10], we discuss the phenomena of asymptotic flatness of the Weil-Petersson resulting from pinching two independent curves on the surface.

When we pinch two nonhomotopic curves γ_0 and γ_1 on surface Σ to two points, say p_0 and p_1, the surface Σ is developing two nodes. We denote M_0 and M_1 pinching neighborhoods for these two geodesics, i.e., two cylinders described as M in §3.1, centered at γ_0 and γ_1, respectively.

We define $M(l_0, l_1)$ be the surface with two of the Fenchel-Nielsen coordinates, namely, the hyperbolic lengths of γ_0 and γ_1, are l_0 and l_1,

respectively. When we set the length of these two geodesics equal to l simultaneously, we will have a point $M(l) = M(l,l)$ in Teichmüller space $\mathcal{T}_g$. As l tends to zero, the surface is developing two nodes. At this point $M(l)$, there are two tangent vectors $i\dot{\mu}_0$ and $\dot{\mu}_1$. Here $\dot{\mu}_0$ is the same as described in §3.2, i.e., we fix γ_1 in M_1 having length l, and pinch γ_0 in M_0 into length $L = L(t)$, where $L(0) = l$. So the t-derivative of $\mu_0(t)$ at $t = 0$ represents a tangent vector, $\dot{\mu}_0$, of Teichmüller space $\mathcal{T}_g$ at $M(l)$, and $i\dot{\mu}_0$ is the tangent vector obtained by performing infinitesimal twists on the curve γ_0, as seen in §3.2. We denote the resulting harmonic map by $W_0(t) : M(l,l) \to M(L(t),l)$. Similarly when we fix γ_0 in M_0 having length l, and pinch γ_1 into length $L = L(t)$, the t-derivative of $\mu_1(t)$ at $t = 0$ represents another tangent vector, $\dot{\mu}_1$, at $M(l)$; we denote the resulting harmonic map by $W_1(t) : M(l,l) \to M(l;L(t))$. We obtain a family of tangent planes Ω'_l, spanned by $i\dot{\mu}_0$ and $\dot{\mu}_1$.

Now we estimate the curvatures of Ω'_l, the result is:

Theorem 1.3. *Ω'_l is asymptotically flat, i.e., its Weil-Petersson sectional curvature is of the order $O(l)$.*

Again, as in §3.2, we will still use rotational symmetric harmonic maps to approximate harmonic maps in the cylinder regions. We describe our "model case two": the surface is a pair of hyperbolic cylinders M_0 and M_1, as we are pinching two independent curves γ_0 and γ_1.

We denote $\phi_0(t)$ as the Hopf differential corresponding to the cylinder map $w_0(t)$ in M_0, and $\phi_1(t)$ as the Hopf differential corresponding to $w_1(t)$ in M_1. Here $w_0(t) : M_0(l) \to M_0(L(t))$ and $w_1(t) : M_1(l) \to M_1(L(t))$ are harmonic maps described as $w = u(x)+iy$ in §3.1. We still use ϕ_0 to denote the Hopf differential corresponding to harmonic map $W_0(t)$ in M_1, and ϕ_1 as the Hopf differential corresponding to harmonic map $W_1(t)$ in M_0.

We still denote $a = a(l) = l^{-1}sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1}sin^{-1}(l)$. Recall from §3.2, we can assume that $|i\dot{\phi}_0| = 1$ in M_0, and also $\dot{\phi}_1 = 1$ in M_1. It is not hard to see that $|\dot{\phi}_1||_{M_0} = \zeta(x,l)$ for $x \in [a,b]$, where $\zeta(x,l)$ satisfies that $\zeta(x,l) \leq C_1 x^{-4}$ for $x \in [a, \pi/2l]$, and $\zeta(x,l) \leq C_1(\pi/l - x)^{-4}$ for $x \in [\pi/2l, b]$, and $\zeta(x,0)$ decays exponentially in $[1,+\infty]$. Here $C_1 = C_1(l)$ is positive and bounded as l tends to zero. To see this, notice that $\dot{\phi}_1$ is holomorphic and $|\dot{\phi}_1|$ is positive, so $log|\dot{\phi}_1|$ is harmonic in the cylinder M_0. Hence we can express $log|\dot{\phi}_1|$ in a Fourier series $\Sigma a_n(x)exp(-iny)$, and we compute $0 = \Delta log|\dot{\phi}_1| = \Sigma(a''_n - n^2 a_n)exp(-iny)$. We will see later on that $\dot{\phi}_1$ is close to 0 in M_0. Hence we conclude the properties ζ has. Similarly, we assume that

$|i\dot{\phi}_0||_{M_1} = \zeta(x,l)$ for $x \in [a,b]$. We see that most of the mass of $|\dot{\phi}_0|$ resides in the thin part associated to γ_0, and most of the mass of $|\dot{\phi}_1|$ resides in the thin part associated to γ_1 ([15]).

Therefore the corresponding Beltrami differentials are:

$$|i\dot{\mu}_0|^2|_{M_0} = |i\dot{\bar{\phi}}_0/\sigma|^2|_{M_0} = l^{-4}sin^4(lx)$$
$$|\dot{\mu}_1|^2|_{M_0} = |\dot{\bar{\phi}}_1/\sigma|^2|_{M_0} = l^{-4}sin^4(lx)\zeta^2(x,l)$$
$$|i\dot{\mu}_0|^2|_{M_1} = l^{-4}sin^4(lx)\zeta^2(x,l)$$
$$|\dot{\mu}_1|^2|_{M_1} = l^{-4}sin^4(lx)$$

4.2 Estimates in Model Case Two

We recall from §2.1, the curvature tensor is given by

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = (\int_\Sigma D(\dot{\mu}_\alpha\dot{\bar{\mu}}_\beta\dot{\mu}_\gamma\dot{\bar{\mu}}_\delta dA) + (\int_\Sigma D(\dot{\mu}_\alpha\dot{\bar{\mu}}_\delta)\dot{\mu}_\gamma\dot{\bar{\mu}}_\beta dA)$$

and the curvature of Ω_l' is R/Π, where

$$R = R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}}$$
$$\Pi = 4 < i\dot{\mu}_0, i\dot{\mu}_0 >^2 < \dot{\mu}_1, \dot{\mu}_1 >^2 - 4| < i\dot{\mu}_0, \dot{\mu}_1 > |^2$$

Now we estimate terms in the model case two where the surface is a pair of cylinders. Similar to Lemma 2 of [10], we have

Lemma 4.1. $1/\Pi = O(l^3)$

Proof. As in §3.2, we have that $< \dot{\mu}_0, \dot{\mu}_0 > |_{M_0} \sim l^{-3}$. And using $|\dot{\mu}_1|^2|_{M_0} = l^{-4}sin^4(lx)\zeta^2(x,l)$, and $\zeta(x,l) \leq C_1 x^{-4}$ for $x \in [a, \pi/2l]$, we have

$$\begin{aligned} < \dot{\mu}_1, \dot{\mu}_1 > |_{M_0} &= \int_{M_0} |\dot{\mu}_1|^2 \sigma dxdy \\ &= 2\int_0^1 \int_a^{\pi/2l} l^{-4}sin^4(lx)\zeta^2(x,l)\sigma dxdy \\ &\leq 2C_1^2 \int_0^1 \int_a^{\pi/2l} l^{-2}sin^2(lx)x^{-8}dxdy \\ &= O(1) \end{aligned}$$

Also $| < i\dot{\mu}_0, \dot{\mu}_1 > | = | \int_\Sigma i\dot{\mu}_0\dot{\bar{\mu}}_1 dA|$, hence,

$$\begin{aligned} | < i\dot{\mu}_0, \dot{\mu}_1 > |_{M_0} &= | \int_{M_0} i\dot{\mu}_0\dot{\mu}_1\sigma dxdy| \\ &\leq C_1 \int_{M_0} l^{-2}sin^2(lx)x^{-4}dxdy \\ &= O(1) \end{aligned}$$

Note that $\Pi \geq (4 < i\dot{\mu}_0, i\dot{\mu}_0 >< \dot{\mu}_1, \dot{\mu}_1 > -4(| < i\dot{\mu}_0, \dot{\mu}_1 > |))^2|_{M_0} \sim l^{-3}$, which completes the proof of Lemma 4.1. $\square$

From Lemma 4.1, we have

$$|R|/\Pi = O(|R|/(l^{-3})) = O(|R|l^3)$$

Now we are left to estimate $|R|$. Similar to Lemma 3 of [10], we have

Lemma 4.2. $|R| \leq 8\int_\Sigma D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy$

Proof. Note that $D = -2(\Delta - 2)^{-1}$ is self-adjoint, hence we have

$$\int_\Sigma D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy = \int_\Sigma D(|\dot{\mu}_1|^2)|i\dot{\mu}_0|^2\sigma dxdy$$

Therefore,

$$\begin{aligned}
R &= R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}} \\
&= 2\int_\Sigma D(i\dot{\mu}_0\dot{\bar{\mu}}_1)i\dot{\mu}_0\dot{\bar{\mu}}_1\sigma dxdy + 2\int_\Sigma D(\dot{\mu}_1(-i\dot{\bar{\mu}}_0))\dot{\mu}_1(-i\dot{\bar{\mu}}_0)\sigma dxdy \\
&- \int_\Sigma D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy - \int_\Sigma D(|\dot{\mu}_1|^2)|i\dot{\mu}_0|^2\sigma dxdy \\
&- \int_\Sigma D(i\dot{\mu}_0\dot{\bar{\mu}}_1)\dot{\mu}_1(-i\dot{\bar{\mu}}_0)\sigma dxdy - \int_\Sigma D(\dot{\mu}_1(-i\dot{\bar{\mu}}_0))i\dot{\mu}_0\dot{\bar{\mu}}_1\sigma dxdy \\
&= -6\int_\Sigma D(\dot{\mu}_0\dot{\mu}_1)\dot{\mu}_1\dot{\mu}_0\sigma dxdy - 2\int_\Sigma D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy
\end{aligned}$$

The last equality follows from that here $\dot{\mu}_0$ and $\dot{\mu}_1$ are real functions. And we have $|D(\dot{\mu}_0\dot{\mu}_1)| \leq |D(|\dot{\mu}_0|^2)|^{1/2}|D(|\dot{\mu}_1|^2)|^{1/2}$ from lemma 4.3 of [25]. This proves Lemma 4.2. $\square$

From Lemmas 4.1 and 4.2, we will need to establish next estimate to prove theorem 1.3:

Lemma 4.3. $\int_\Sigma D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy = O(l^{-2})$

Proof. The surface is a pair of two cylinders, so we need to estimates two integrals: $\int_{M_0} D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA$ and $\int_{M_1} D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA$.

For the first integral $\int_{M_0} D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA$, we write $D(|\dot{\mu}_0|^2) = \ddot{\mathcal{H}}$, where $\mathcal{H}(t)$ is the holomorphic energy for the family of harmonic maps $w_0(t)$, as in §3.2. Then in M_0, we have $(\Delta - 2)\ddot{\mathcal{H}} = -2\frac{|\dot{\phi}_0|^2}{\sigma^2} = -2l^{-4}sin^4(lx)$. We recall that $\ddot{\mathcal{H}}$ is bounded on the compacta in $[a, b]$

and so we can assume that $A_1(l) = \ddot{\mathcal{H}}(a) = \ddot{\mathcal{H}}(l^{-1}sin^{-1}l) = O(1) > 0$. Then $\ddot{\mathcal{H}}(x)$ solves the differential equation

$$(l^{-2}sin^2(lx))\ddot{\mathcal{H}}'' - 2\ddot{\mathcal{H}} = -2l^{-4}sin^4(lx)$$

with the conditions

$$\ddot{\mathcal{H}}(l^{-1}sin^{-1}l) = A_1(l), \ddot{\mathcal{H}}'(\pi/2l) = 0$$

We remark here that infinitesimal twists only change the boundary values for $\ddot{\mathcal{H}}$, which will stay bounded.

We have solved above differential equation in §3.2, and the general solutions have the following form

$$\ddot{\mathcal{H}}(l;x) = \tfrac{sin^2(lx)}{2l^4} + A_2cot(lx) + A_3(1 - lxcot(lx))$$

where coefficients $A_2 = A_2(l)$ and $A_3 = A_3(l)$ are constants that satisfy

$$A_2 = \tfrac{\pi}{2}A_3 = O(l^{-1})$$

Noticing that $\zeta(x,l) \le C_1x^{-4}$ in $[a, \pi/2l]$, we compute the first integral:

$$\begin{aligned}
\int_{M_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy &= \int_{M_0} \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2\sigma dxdy \\
&= \int_0^1\int_a^b \ddot{\mathcal{H}}(x)(l^{-2}sin^2(lx))\zeta^2(x,l)dxdy \\
&\le 2C_1^2\Big(\int_a^{\pi/2l} \frac{sin^2(lx)}{2l^4}l^{-2}sin^2(lx)x^{-8}dx \\
&+ \int_a^{\pi/2l} A_2cot(lx)l^{-2}sin^2(lx)x^{-8}dx \\
&+ \int_a^{\pi/2l} A_3(1 - lxcot(lx))l^{-2}sin^2(lx)x^{-8}dx\Big) \\
&= O(l^{-2}) + O(l^{-2}) + O(l^{-2}) \\
&= O(l^{-2}) \qquad (7)
\end{aligned}$$

Now let us look at the second integral $\int_{M_1} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy$. Note that in M_1, which we identify with $[a,b] \times [0,1]$, we have $D(|\dot{\mu}_0|^2) = \ddot{\mathcal{H}}$, here $\mu_0(t)$ and $\mathcal{H}$ come from the harmonic map $W_0(t) : M(l,l) \to M(L(t),l)$, where $|\dot{\mu}_0||_{M_1} = l^{-2}sin^2(lx)\zeta(x,l)$, and $\mathcal{H}(t)$ solves

$$(l^{-2}sin^2(lx))\ddot{\mathcal{H}}'' - 2\ddot{\mathcal{H}} = -2l^{-4}sin^4(lx)\zeta^2(x,l) \qquad (8)$$

with the conditions

$$\ddot{\mathcal{H}}(l^{-1}sin^{-1}l) = B_1(l),\ \ddot{\mathcal{H}}(\pi/l - l^{-1}sin^{-1}l) = B_2(l).$$

Here $B_1(l)$ and $B_2(l)$ are positive and bounded as l tends to zero, since $\ddot{\mathcal{H}}$ converges to the holomorphic energy in the "noded" problem (of §3.2, when $M_1 = [1,\infty)$). We recall that $|\dot{\phi}_1||_{M_0} = \zeta(x,l)$, where $\zeta(x,l) \le C_1 x^{-4}$ for $x \in [a, \pi/2l]$ and $\zeta(x,l) \le C_1(\pi/l - x)^{-4}$ for $x \in [\pi/2l, b]$.

Consider the equation

$$(l^{-2}sin^2(lx))Y'' - 2Y = 0 \tag{9}$$

with the boundary conditions that satisfy

$$Y(l^{-1}sin^{-1}l) = O(1) \text{ and } Y(\pi/l - l^{-1}sin^{-1}l) = O(1), \text{ as } l \to 0.$$

We choose $h(x) = -x^{-2}$ and claim that $\ddot{\mathcal{H}} - h$ is a subsolution to (9) for $x \in [a,b]$. To see this, noticing that $2|\dot{\phi}_0|^2|_{M_1} l^{-4} sin^4(lx)$ decays rapidly as $x \to b$ for small l, we have

$$(l^{-2}sin^2(lx))(\ddot{\mathcal{H}} - h(x))'' - 2(\ddot{\mathcal{H}} - h(x)) = (\frac{6sin^2(lx)}{l^2x^4} - \frac{2}{x^2}) - 2|\dot{\mu}_0|^2 > 0 \tag{10}$$

Notice that $\lambda Y(x)$ solves the equation (9) if $Y(x)$ does, for any constant λ. So up to multiplying by a bounded constant, we have $Y|_{\partial M_1} > (\ddot{\mathcal{H}} - h(x))|_{\partial M_1}$. Hence $\ddot{\mathcal{H}} - h$ is a subsolution to (9) for $x \in [a,b]$, while the solutions to (9) have the form

$$Y(l;x) = B_3 cot(lx) + B_4(1 - lx cot(lx))$$

where constants $B_3 = B_3(l)$ and $B_4 = B_4(l)$ satisfy, from the boundary conditions for the equation (9), that

$$B_3 = O(l) \text{ and } B_4 = O(l)$$

Therefore in $[a,b]$, We have $\ddot{\mathcal{H}} \le h(x) + Y(x)$. Now,

$$\begin{aligned}
\int_{M_1} Y(x)|\dot{\mu}_1|^2 \sigma dxdy &= \int_0^1 \int_a^b Y(x)(l^{-2}sin^2(lx))dxdy \\
&= \int_a^b B_3 cot(lx)(l^{-2}sin^2(lx))dx \\
&+ \int_a^b B_4(1 - lx cot(lx))(l^{-2}sin^2(lx))dx \\
&= O(l^{-2}) + O(l^{-2}) \\
&= O(l^{-2})
\end{aligned}$$

Now we compute the second integral:

$$
\begin{aligned}
\int_{M_1} D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA &= \int_0^1 \int_a^b \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 dA \\
&\leq \int_0^1 \int_a^b (Y(x)+h(x))|\dot{\mu}_1|^2 \sigma dx dy \\
&= O(l^{-2}) + O(l^{-1}) = O(l^{-2}). \qquad (11)
\end{aligned}
$$

This proves Lemma 4.3. $\square$

Combined with the estimates of Lemmas 4.1, 4.2 and 4.3, we proved theorem 1.3 when the surface is a pair of cylinders.

4.3 Proof of Theorems 1.3 and 1.4

We are now ready to complete the proof of theorem 1.3. Recall from §3.3, we constructed a family of maps $G_0(t)$ to approximate $W_0(t)$. The map $G_0(t)$ consists of three parts: the cylinder map w_0 of M_0 deep into the cylinder region in $M_0 \cap \Sigma_0$, the identity map in the compact region $\Sigma_0 \backslash M_0$, and a smooth map interplaying with the identity map and w_0 in the intersection region $B(\partial M_0, 1)$. Similarly, we obtain a family of maps $G_1(t)$ such that it is w_1 in $M_1 \cap \Sigma_1$, the identity map in $\Sigma_1 \backslash M_1$, and a smooth map in $B(\partial M_1, 1)$. Here $\Sigma_0 = \{p \in \Sigma : dist(p, \partial M_0) > 1\}$, and $B(\partial M_0, 1) = \{p \in \Sigma : dist(p, \partial M_0) \leq 1\}$.

Proof of Theorem 1.3. Parallel to the discussion in §3.3, we know that both families $G_0(t)$ and $G_1(t)$ are close to the families of harmonic maps $W_0(t)$ and $W_1(t)$, respectively (Lemma 3.4). Also $|\dot{\phi}_{G_0} - \dot{\phi}_{W_0}| = O(1)$ and $|\dot{\phi}_{G_1} - \dot{\phi}_{W_1}| = O(1)$ both hold for $x \in [a+1, b-1]$ (Lemma 3.6).

It is not hard to see that Lemma 4.1 still holds, so we need to establish the same estimate as in Lemma 4.3 in order to prove theorem 1.3 in the general case, i.e., we need to estimate $\int_\Sigma D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy$,

which breaks into three integrals as following:

$$
\begin{aligned}
\int_{\Sigma} D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy &= \int_{M_0\cap\Sigma_0} D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy \\
&+ \int_{M_1\cap\Sigma_1} D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy \\
&+ \int_{\mathcal{K}} D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2\sigma dxdy \\
&= \int_{M_0\cap\Sigma_0} \ddot{\mathcal{H}}|\dot{\mu}_1|^2\sigma dxdy \\
&+ \int_{M_1\cap\Sigma_1} \ddot{\mathcal{H}}|\dot{\mu}_1|^2\sigma dxdy + O(1) \qquad (12)
\end{aligned}
$$

where $\mathcal{K}$ is the compact set disjoint from $(M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)$.

Notice that $\partial\mathcal{K} = \partial((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1))$, so parallel to the discussion in §3.4, the maximum principle will force that $\ddot{\mathcal{H}} \leq \lambda\ddot{\mathcal{H}}^G + Y(l,x)$ in $(M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)$, for some bounded constant λ, where $Y(l,x) = B_3cot(lx) + B_4(1 - lxcot(lx))$ is the solution to $(\Delta - 2)Y = 0$, and constants B_3 and B_4 satisfy that $B_3 = O(l)$ and $B_4 = O(l)$. Apply this to (12):

$$
\begin{aligned}
\int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA &= \int_{(M_0\cap\Sigma_0)\cup(M_1\cap\Sigma_1)} \ddot{\mathcal{H}}|\dot{\mu}_1|^2 dA + O(1) \\
&\leq \int_{(M_0\cap\Sigma_0)} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(|\dot{\mu}_1|^2)dA \\
&+ \int_{(M_1\cap\Sigma_1)} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(|\dot{\mu}_1|^2)dA + O(1) \\
&\leq \int_{M_0} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(2|\dot{\mu}_1|^2 + 2|\dot{\mu}_1 - \dot{\mu}_1^G|^2)dA \\
&+ \int_{M_1} (\lambda\ddot{\mathcal{H}}^G + Y(l;x))(2|\dot{\mu}_1|^2 + 2|\dot{\mu}_1 - \dot{\mu}_1^G|^2)dA \\
&+ O(1)
\end{aligned}
$$

From the calculation in §3.3, and $|\dot{\mu}_1 - \dot{\mu}_1^G| = |\dot{\phi}_1 - \dot{\phi}_1^G|/\sigma$, where

$|\dot{\phi}_1 - \dot{\phi}_1^G| \le C_5 l^2 x^{-2} csc^2(lx)$ for $x \in [a+1, \pi/2l]$, we have the following:

$$\begin{aligned}
\int_{M_0} (\lambda \ddot{\mathcal{H}}^G)|\dot{\mu}_1^G|^2 dA &= O(l^{-2}) \\
\int_{M_1} (\lambda \ddot{\mathcal{H}}^G)|\dot{\mu}_1^G|^2 \sigma dxdy &= O(l^{-2}) \\
\int_{M_0} Y(l,x)|\dot{\mu}_1^G|^2 \sigma dxdy &= O(1) \\
\int_{M_1} Y(l,x)|\dot{\mu}_1^G|^2 \sigma dxdy &= O(l^{-2}) \\
\int_{M_0 \cap \Sigma_0} (\ddot{\mathcal{H}}^G + Y(l;x))(|\dot{\mu}_1 - \dot{\mu}_1^G|^2) dA &= o(l^{-2}) \\
\int_{M_1 \cap \Sigma_1} (\ddot{\mathcal{H}}^G + Y(l;x))(|\dot{\mu}_1 - \dot{\mu}_1^G|^2) dA &= o(l^{-2})
\end{aligned}$$

Applying these estimates to (12), we have

$$\int_\Sigma D(|i\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA = O(l^{-2}).$$

This and Lemma 4.1 complete the proof of theorem 1.3. □

To prove theorem 1.4, we consider the plane Ω_l'', spanned by Beltrami differentials $i\dot{\mu}_0$ and $i\dot{\mu}_1$. Again we find that Lemma 4.1 still holds, i.e., $1/\Pi = O(l^3)$, where $\Pi = 4 < i\dot{\mu}_0, i\dot{\mu}_0 >^2 < i\dot{\mu}_1, i\dot{\mu}_1 >^2 - 4 | < i\dot{\mu}_0, i\dot{\mu}_1 > |^2$. A direct calculation shows that R is given by

$$\begin{aligned}
R &= R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}} \\
&= 2\int_\Sigma D(i\dot{\mu}_0(-i)\dot{\bar{\mu}}_1)\dot{\mu}_0\dot{\bar{\mu}}_1 dA + 2\int_\Sigma D(i\dot{\mu}_1(-i\dot{\bar{\mu}}_0))\dot{\mu}_1(\dot{\bar{\mu}}_0) dA \\
&- \int_\Sigma D(|i\dot{\mu}_0|^2)|i\dot{\mu}_1|^2 dA - \int_\Sigma D(|i\dot{\mu}_1|^2)|i\dot{\mu}_0|^2 dA \\
&- \int_\Sigma D(i\dot{\mu}_0(-i)\dot{\bar{\mu}}_1) i\dot{\mu}_1(-i\dot{\bar{\mu}}_0) dA - \int_\Sigma D(i\dot{\mu}_1(-i\dot{\bar{\mu}}_0)) i\dot{\mu}_0(-i)\dot{\bar{\mu}}_1 dA \\
&= 2\int_\Sigma D(\dot{\mu}_0\dot{\mu}_1)\dot{\mu}_1\dot{\mu}_0 dA - 2\int_\Sigma D(|i\dot{\mu}_0|^2)|i\dot{\mu}_1|^2 dA
\end{aligned}$$

Hence we have a result similar to that of Lemma 4.2:

Lemma 4.4. $|R| \le 4\int_\Sigma D(|i\dot{\mu}_0|^2)|i\dot{\mu}_1|^2 dA$

Now the curvature of Ω_l''' is R/Π. An argument parallel to that of proving theorem 1.3 leads to a proof of theorem 1.4. □

5 Asymptotic Flatness II: Curvature Bounds

The goal of this section is to prove theorem 1.5 and, finally, theorem 1.1. We continue to discuss the asymptotic flatness of the Weil-Petersson metric when there is only one shrinking geodesic on the surface in §5.1, then give a proof of theorem 1.1 in §5.2.

In this section, we always use l to denote the length of the shortest geodesic on the surface.

5.1 One Curve Pinching

In this subsection, we assume that Σ is a closed surface of genus g at least two, and γ_0 is a closed separating short geodesic on the surface Σ with length l. As l tends to zero, the surface is developing a separating node. Let γ_2 and γ_3 be two closed geodesics on the different sides of curve γ_0. Without loss of generality, we assume $l(\gamma_2) = l(\gamma_3) >> l$.

As in previous sections, we denote $M(l, \theta_2, \theta_3)$ as the surface with three of the Fenchel-Nelsen coordinates, the hyperbolic length of γ_0, the twisting angles of γ_2 and γ_3, being l, θ_2, θ_3, respectively.

We denote M_0 as a cylinder centered at γ_0, while M_2 and M_3 are cylinders centered at γ_2 and γ_3, respectively. Note that as the length of γ_0 is kept very short, the cylinder M_0 becomes very long and similar to the cylinder described as M in §3.1. Now we have two families of harmonic maps: the family $W_2(t) : M(l) \to M(l, \theta_2(t), 0)$, which fixes γ_3, keeps γ_0 very short at length l, and twists γ_2 at angle $\theta_2(t)$; and the family $W_3(t) : M(l) \to M(l, \theta_3(t), 0)$, which fixes γ_2, keeps γ_0 very short at length l, and twists γ_3 at angle $\theta_3(t)$. We denote $\dot{\mu}_2$ and $\dot{\mu}_3$ as infinitesimal Beltrami differentials corresponding to $W_2(t)$ and $W_3(t)$, respectively. Also $\dot{\phi}_2$ and $\dot{\phi}_3$ are corresponding infinitesimal Hopf differentials. Now we obtain a tangent plane Ω_l''' at a point $M(l) = M(l, 0, 0)$ in Teichmüller space, spanned by tangent vectors $\dot{\mu}_2$ and $\dot{\mu}_3$, and we will show that

Theorem 1.5. *This plane Ω_l''' is asymptotically flat with respect to the Weil-Petersson metric, moreover, its Weil-Petersson sectional curvature is of the order $O(l)$.*

The curvature of Ω_l''' is given by R/Π, where $R = R_{2\bar{3}2\bar{3}} - R_{2\bar{3}3\bar{2}} - R_{3\bar{2}2\bar{3}} + R_{3\bar{2}3\bar{2}}$ and $\Pi = 4 < \dot{\mu}_2, \dot{\mu}_2 >< \dot{\mu}_3, \dot{\mu}_3 > -2| < \dot{\mu}_2, \dot{\mu}_3 > |^2 - 2Re(< \dot{\mu}_2, \dot{\mu}_3 >)^2$. Here we recall that the curvature tensor is

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = (\int_\Sigma D(\dot{\mu}_\alpha \dot{\bar{\mu}}_\beta \dot{\mu}_\gamma \dot{\bar{\mu}}_\delta dA) + (\int_\Sigma D(\dot{\mu}_\alpha \dot{\bar{\mu}}_\delta) \dot{\mu}_\gamma \dot{\bar{\mu}}_\beta dA)$$

The essential parts on the surface for the curvature estimates are M_0, M_2 and M_3. As in previous sections, we will use rotationally symmetric harmonic maps $w_i(t)$ to approximate $W_i(t)$ in M_i, for $i = 2, 3$. We still denote the Hopf differentials of $w_i(t)$ by $\phi_i(t)$ and corresponding Beltrami differentials by $\mu_i(t)$ for $i = 2, 3$.

Similar to the discussion in model cases one and two, the infinitesimal Hopf differentials $\dot{\phi}_2(t)$ and $\dot{\phi}_3(t)$ are holomorphic. Since harmonic maps $w_2(t)$ and $w_3(t)$ are rotationally symmetric, we can assume the infinitesimal Hopf differentials $|\dot{\phi}_2|_{M_2} = 1$, $|\dot{\phi}_3|_{M_3} = 1$, while $|\dot{\phi}_2|_{M_0} = \eta_2(x, l)$, and $|\dot{\phi}_3|_{M_0} = \eta_3(x, l)$, where we consider M_0 as a cylinder characterized by $[a, b] \times [0, 1]$, and $a = a(l) = l^{-1}sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1}sin^{-1}(l)$. Here, similar to $\zeta(x, l)$ from §4.1, the functions $\eta_2(x, l)$ and $\eta_3(x, l)$ satisfy that $\eta_2(x, l) \leq C_1 x^{-4}$ for $x \in [a, b]$, and $\eta_3(x, l) \leq C_1(b - x)^{-4}$ for $x \in [a, b]$, and they decay exponentially in $[1, +\infty]$. We note that, asymptotically, cylinders M_2 and M_3 lie on different sides of M_0. Therefore we can assume that $|\dot{\phi}_2|_{M_3} = O(l^4)$ and $|\dot{\phi}_3|_{M_2} = O(l^4)$.

To prove theorem 1.5, we want to show firstly that Π is bounded (away from zero). This is easy to see as, asymptotically, both $\dot{\phi}_2(t)$ and $\dot{\phi}_3(t)$ decay exponentially in M_0, and hence $\Pi > \Pi|_{M_0}$ is bounded from below by some positive constant. Now we just have to show that $|R| = O(l)$.

Lemma 5.1. $\int_{M_0} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2 \sigma dxdy = O(l)$

Proof. We split the interval $[a, b]$ into three parts: $[a, l^{-1/4}]$, $[l^{-1/4}, b - l^{-1/4}]$ and $[b - l^{-1/4}, b]$. Recall that $D(|\dot{\mu}_2|^2) = \ddot{\mathcal{H}}^M$, where $\ddot{\mathcal{H}}^M$ is the holomorphic energy corresponding to the family of harmonic maps $w_2(t)$ resulting from twisting the curve γ_2.

Similar to the proof of Lemma 4.3, we compare $\ddot{\mathcal{H}}^M$ with $Y(l; x)$, where $Y(l, x) = B_3 cot(lx) + B_4(1 - lxcot(lx))$ is the solution to $(\Delta - 2)Y = 0$ in $[a, b]$ with constants B_3 and B_4 satisfying that $B_3 = O(l)$ and $B_4 = O(l)$. Again, we find that $\ddot{\mathcal{H}} + x^{-2}$ is a subsolution to $(\Delta - 2)Y = 0$ for $x \in [a, b]$. Here the hyperbolic length element in M_0 is $lcsc(lx)|dz|$.

In the first interval $[a, l^{-1/4}]$, noticing that $|\dot{\phi}_3| \leq C_1(b - l^{-1/4})^{-4} =$

$O(l^4)$, we have

$$\begin{aligned}\int_a^{l^{-1/4}} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2\sigma dx &\leq \int_a^{l^{-1/4}} (Y(x)-x^{-2})|\dot{\mu}_3|^2\sigma dx \\ &= \int_a^{l^{-1/4}} (Y(x)-x^{-2})|\dot{\phi}_3|^2 l^{-2} sin^2(lx)dx \\ &= o(l)\end{aligned}$$

In the second interval $[l^{-1/4}, b-l^{-1/4}]$, we have

$$\begin{aligned}\int_{l^{-1/4}}^{b-l^{-1/4}} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2\sigma dx &\leq \int_{l^{-1/4}}^{b-l^{-1/4}} (Y(x)-x^{-2})|\dot{\mu}_3|^2\sigma dx \\ &\leq \int_{l^{-1/4}}^{b-l^{-1/4}} \frac{(Y(x)-x^{-2})C_1^2}{(b-x)^8} l^{-2} sin^2(lx)dx \\ &= o(l)\end{aligned}$$

In the third interval $[b-l^{-1/4}, b]$, noticing that $D(|\dot{\mu}_2|^2) \leq Y(x)-x^{-2} = O(l)$, we have

$$\begin{aligned}\int_{b-l^{-1/4}}^{b} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2\sigma dx &\leq \int_{b-l^{-1/4}}^{b} (Y(x)-x^{-2})|\dot{\mu}_3|^2\sigma dx \\ &= \int_{b-l^{-1/4}}^{b} O(l)\frac{C_1^2}{(b-x)^8} l^{-2} sin^2(lx)dx \\ &= O(l)\end{aligned}$$

Combining these calculation, we complete the proof of Lemma 5.1. □

To show $|R| = O(l)$, we notice that $|R| \leq C' \int_\Sigma D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2\sigma dxdy$ for some positive constant C', as in the proof of Lemma 3 of [10] and Lemma 4.2. Hence we need to calculate the integral $\int_\Sigma D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2 dA$. Since γ_0 is a separating curve, we split this integral into two parts, one for each side of γ_0, and we abuse our notation to denote the side with curve γ_2 by M_2, and the other side with curve γ_3 by M_3.

Since M_2 is compact, and $|\dot{\phi}_3|_{M_2} = O(l^4)$, we have the integral

$$\int_{M_2} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2\sigma dxdy = O(l^4) = o(l) \tag{13}$$

Also M_3 is compact, we have $|\dot{\phi}_3|_{M_3} = O(1)$, while $D(|\dot{\mu}_2|^2)|_{M_3} = \ddot{\mathcal{H}^{\mathcal{M}}}|_{M_3}$ is comparable to $\ddot{\mathcal{H}^{\mathcal{M}}}|_{M_0\cap M_3}$, which is of the order $O(lcsc(lb)) =$

$O(1/b) = O(l)$, recalling that $b = b(l) = \pi l^{-1} - l^{-1} sin^{-1}(l)$. Thus

$$\int_{M_3} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2 \sigma dxdy = O(l) \tag{14}$$

The estimate of Lemma 5.1, (13) and (14) complete the proof of theorem 1.5 when we consider the family of harmonic maps being rotationally symmetric.

Now we consider the families of harmonic maps which are no longer rotationally symmetric. Since γ_0 separates the surface, the cylinder M_0 separates the surface into two sides. We denote the side containing γ_2 by M_2, and the side containing γ_3 by M_3. As we indicated, the infinitesimal Hopf differentials are exponentially small in the thin part. Thus we can still assume that $|\dot{\phi}_2|_{M_3} = O(l^4)$ and $|\dot{\phi}_3|_{M_2} = O(l^4)$, hence the term Π in the curvature formula is still bounded.

To compute the integral $\int_\Sigma D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2 dA$, we still denote the holomorphic energy of the family $W_2(t)$ by $\mathcal{H}(t)$, then $D(|\dot{\mu}_2|^2) = \ddot{\mathcal{H}}$ will be dominated, as seen in the argument in §3.4 after (6), by $\ddot{\mathcal{H}^{\mathcal{M}}}$ and $Y(l,x) = B_3 cot(lx) + B_4(1 - lxcot(lx))$, the solution to $(\Delta - 2)Y = 0$ in $[a,b]$ with constants B_3 and B_4 satisfying that $B_3 = O(l)$ and $B_4 = O(l)$, when we characterize M_0 by $[a,b] \times [0,1]$.

Now it is not hard to see that

$$\begin{gathered} \int_{\Sigma_2 \cup \Sigma_3} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2 dA = o(l) \\ \int_{[a,b-l^{1/4}] \times [0,1]} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2 dA = o(l) \end{gathered}$$

While $\int_{[b-l^{1/4},b] \times [0,1]} D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2 dA = O(l)$ as $Y(l,x) \sim lcot(lx) \sim l$ when $x \in [b - l^{1/4}, b]$. Therefore the integral $\int_\Sigma D(|\dot{\mu}_2|^2)|\dot{\mu}_3|^2 dA = O(l)$ and we complete the proof of theorem 1.5. □

Remark 5.2. *If the shrinking curve $\gamma_0(l)$ is not separating, the twisting neighborhoods M_2 and M_3 will lie on the same component in the limit of surfaces with one shrinking curve as l tends to zero. It is not hard to see that $K(\Omega_l''')$ is now bounded away from zero.*

Remark 5.3. *Theorem 1.5 can be easily generalized to treat the case when multiple closed geodesics are pinching and at least one of them is separating. Infinitesimal twists about curves on different sides of a separating and shrinking curve will give a family of asymptotically flat tangent planes.*

5.2 Curvature Bounds

In this subsection, we prove theorem 1.1, i.e., we give curvature bounds at any point in Teichmüller space.

Theorem 1.1. *Let l be the length of the shortest geodesic on closed surface Σ, and K be the Weil-Petersson sectional curvature of Teichmüller space $\mathcal{T}$, assuming $dim_C\mathcal{T} > 1$, there exists a constant $C > 0$ such that*

$$-(Cl)^{-1} \leq K \leq -Cl$$

Moreover, there are tangent planes with the Weil-Petersson curvatures of the orders $O(l)$ and comparable to l^{-1}, and hence the Weil-Petersson sectional curvature has neither negative upper bound, nor lower bound.

Recall from §3.4, we showed that the Weil-Petersson holomorphic sectional curvature tends to negative infinity at the rate of the order $O(l^{-1})$. We note that, asymptotically, the absolute value of the sectional curvature is dominated by diagonal terms. One way to see this, assume that $\dot{\nu}_0$ and $\dot{\nu}_1$ are two tangent vectors at Σ in the tangent space of Teichmüller space, then, as in lemma 4.3 of [25], we have $|D(\dot{\nu}_0\bar{\dot{\nu}}_1)| \leq D(|\dot{\nu}_0|^2)^{1/2}D(|\dot{\nu}_1|^2)^{1/2}$. Applying Schwarz lemma, one finds that the absolute value of the curvature of the plane, spanned by $\dot{\nu}_0$ and $\dot{\nu}_1$, is dominated by $\int_\Sigma D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2dA/\Pi$, where $\Pi = 4 < \dot{\nu}_0, \dot{\nu}_0 >< \dot{\nu}_1, \dot{\nu}_1 > -2| < \dot{\nu}_0, \dot{\nu}_1 > |^2 - 2Re(< \dot{\nu}_0, \dot{\nu}_1 >)^2$.

If there is a lower bound for the length of the shortest closed geodesics on Σ, the Weil-Petersson sectional curvature of Teichmüller space is bounded since all integrals in curvature terms are bounded away from negative infinity and zero. Hence, to consider large absolute value of the sectional curvature, we can assume one of the vectors $\dot{\nu}_0$ and $\dot{\nu}_1$ is corresponding to a deformation of the length of a short geodesic on the surface. Recalling that the Hopf differential not corresonding to pinching this curve will be exponentially small in the thin part of this shrinking curve, so above integral $\int_\Sigma D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2dA$ will be no more than the integral $\int_\Sigma D(|\dot{\nu}_0|^2)|i\dot{\nu}_0|^2dA$, which we estimated in the proof of theorem 1.2. Therefore the proof of theorem 1.2 implies the following theorem, which is the lower bound part of theorem 1.1, i.e.,

Theorem 5.4. *Let l be the length of the shortest closed geodesic on the surface, then there is a positive constant C such that the Weil-Petersson sectional curvature K of Teichmüller space satisfies $K \geq -(Cl)^{-1}$, if the complex dimension of Teichmüller space is great than one.*

The estimates in proving theorem 1.2 immediately imply the following result of Georg Schmuacher:

Corollary 5.5. *([20]) The sectional, Ricci and scalar curvature are asymptotically bounded by* $\sum_{i=1}^{q} log|t_i|$.

Note that since the Weil-Petersson sectional curvature is negative, therefore this theorem of Georg Schmuacher doesnot imply that the curvature is not bounded from below.

To show the upper bound part of theorem 1.1, we notice that the upper bounds for asymptotically flat tangent planes Ω_l in theorem 1 of [10], planes Ω_l' in theorem 1.3, planes Ω_l'' in theorem 1.4, and planes Ω_l''' in theorem 1.5 are all of the order $O(l)$. We want to show $O(l)$ is the right order, in other words, we need to show

Theorem 5.6. *If* $dim_C \mathcal{T} > 1$*, and* l *is the length of the shortest geodesics on the surface, there exists a constant* $C > 0$ *such that the Weil-Petersson sectional curvature* K *of Teichmüller space satisfies* $K \leq -Cl$*.*

We assume at least one core geodesics is shrinking on the surface. Let $(l_1, \theta_1, l_2, \theta_2, ..., l_{3g-3}, \theta_{3g-3})$ be the Fenchel-Nielsen coordinates at a point Σ in Teichmüller space. It suffices to prove theorem 5.6 by assuming two tangent vectors $\dot{\nu}_0$ and $\dot{\nu}_1$ are infinitesimal Beltrami differentials rsulting from deformation of lengths of core geodesics or deformation of twisting angles or both.

From remark 5.2, if both $\dot{\nu}_0$ and $\dot{\nu}_1$ are resulting from deformations of twisting angles about independent core geodesics, then this shrinking curve, denoted by γ with length l, can be assumed to be separating (in such case, we require the genus of the surface is at least two). Therefore, from the proof of theorem 1.5, remark 5.2 and remark 5.3, we have $K(Span(\dot{\nu}_0, \dot{\nu}_1)) = O(l)$.

Now we assume at least one of $\dot{\nu}_0$ and $\dot{\nu}_1$ is resulting from a deformation of the length of a core geodesic on the surface, with or without twisting about this geodesic. Since operator $D = -2(\Delta - 2)^{-1}$ is self-adjoint, we assume $\dot{\nu}_0$ is resulting from a deformation of the length of a core geodesic γ_0. As in the argument before theorem 5.4, we need to estimate the integral $\int_\Sigma D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA$ as it will dominate the absolute value of $R = R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}}$.

Let M_0 be the pinching neighborhood of the shrinking geodesic γ_0 with $l(\gamma_0) = l$, as before, we can characterize M_0 as $[a, b] \times [0, 1]$, where

$a = a(l) = l^{-1}sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1}sin^{-1}(l)$. Therefore we find that $D(|\dot{\nu}_0|^2)$ is of the order of $O(lcot(lb)) = O(l)$ in $\Sigma \backslash M_0$. Hence,

$$\begin{aligned}\int_\Sigma D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA &= \int_{M_0} D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA + \int_{\Sigma\backslash M_0} D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA \\ &= \int_{M_0} D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA + O(l)\int_{\Sigma\backslash M_0} |\dot{\nu}_1|^2 dA\end{aligned}$$

Note that $\int_{\Sigma\backslash M_0} |\dot{\nu}_0|^2 dA = O(1)$, we have the absolute value of the curvature is given by

$$\begin{aligned}|R|/\Pi &\leq \frac{C' \int_\Sigma D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA}{\Pi} \\ &\leq C' \frac{\int_{M_0} D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA}{\Pi} + O(l)\frac{\int_{\Sigma\backslash M_0} |\dot{\nu}_1|^2 dA}{\Pi} \\ &= C'(\frac{\int_{M_0} D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA}{\Pi}) + O(l)\end{aligned}$$

We notice that the integral $\int_{M_0} D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA$ is positive, and will be smaller when the infinitesimal Beltrami differentials $\dot{\nu}_1$ is resulting from a deformation of the length of a core geodesic, independent of γ_0, on the surface, than otherwise. To see this, we recall that $|\dot{\phi}_1|$, where $\dot{\phi}_1$ is the infinitesimal Hopf differential corresponding to $\dot{\nu}_1$, decays exponentially away from the curve where the deformation occurs. Therefore $|\dot{\phi}_1|$ is smaller in M_0 when it is actually resulting from a deformation of the length of a core geodesic γ_1 than the case of otherwise. Now the proofs of theorem 1 of [10], theorem 1.3, and theorem 1.4 imply that

$$\begin{aligned}|R|/\Pi &\leq C'(\frac{\int_{M_0} D(|\dot{\nu}_0|^2)|\dot{\nu}_1|^2 dA}{\Pi}) + O(l) \\ &= O(l) + O(l) = O(l)\end{aligned}$$

This completes the proof of theorem 5.6. Theorem 1.1 immediately follows from theorem 5.4 and theorem 5.6.

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Zheng Huang
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
email address: zhengh@umich.edu